13. D. H. Payne and W. A. Cambleng, Ceram. Bull,. 55, No. 2, 195-197 (1976).
14. A. Gauronnek and A. Regreny, Fiber Drawing Technology: 2 National Symp. Jaglone, Poland (1976), pp. 19-37.
15. M. Nakahara, S. Sakaguchi, and T. Munyashita, Rev. Electr. Commun. Lab., 26, No. 3-4, 476-483 (1978).
16. E. M. Dianov, et al., Pis'ma Zh. Tekh. Fiz., 11, No. 8, 473 (1985).

ANALYSIS OF THE ACCURACY OF THE SOLUTION OF A BOUNDARY-VALUE PROBLEM ON THE BASIS OF A NUMERICAL INVERSION OF THE LAPLACE TRANSFORM

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A study is made of the accuracy of the solution of a problem concerned with the cooling of a semibounded body on the basis of the numerical inversion of the Laplace transform. The exacct and perturbed values of the image function are used as initial data.

An effective method for solving problems of heat and mass transport involves use of the Laplace transform and its subsequent inversion. The inversion problem is ill-posed (see, for example, [1, 2]) in the sense that a small change in the image-function can give rise to a large change in the original function. Often, a solution, expressed in terms of Laplace transforms, is such that it is not possible to obtain an analytical description of the result (by virtue, for example, of the transcendental nature of the expressions involved), a situation which entails the application of numerical methods for the inversion. In solving differential equations of parabolic type it is possible to apply an inversion algorithm [3], whereby one seeks the original of a function in accordance with the expression

$$F(x)=\frac{\ln 2}{x}\sum_{n=1}^{N} V_n f\left(\frac{n}{x}\ln 2\right),$$

$$V_n=(-1)^{n+N/2}\sum_{m=L}^{M}\frac{m^{N/2}(2m)!}{\left(\frac{N}{2}-m\right)!\,m!\,(m-1)!\,(n-m)!\,(2m-n)!}, \tag{1}$$

$$L=(n+1)/2,\ M=\min(n,\ N/2).$$

The algorithm was tested on a number of tabular functions arising in heat and mass-transfer problmes; its application constituted an effective means for recovery of the original function. A comparison of the results obtained with calculations based on a known analytical representation of the function showed agreement to six significant digits. It proved convenient here to choose the number N of basis functions and, correspondingly, the number of terms in formula (1) equal to 10.

In working with systems subject to the action of random disturbances, and also in obtaining the values of functions which are the result of one or another approximating algorithm, it is important to have information concerning the noise stability of the procedure used [4]. Therefore we conducted a check on the stability of the Haver–Stefest procedure [3] used with the aid of a stochastic amplitude modulation of the image-function. Essentially, our check amounted to adding to the value of the image-function found, the latter being given analytically, a random quantity corresponding to a pattern error of the numerical algorithm. In carrying out our numerical experiments we used standard random number generators (with a uniform distribution over the interval $(-\sqrt{3},\ \sqrt{3})$ and with a normal distribution with parameters (0,

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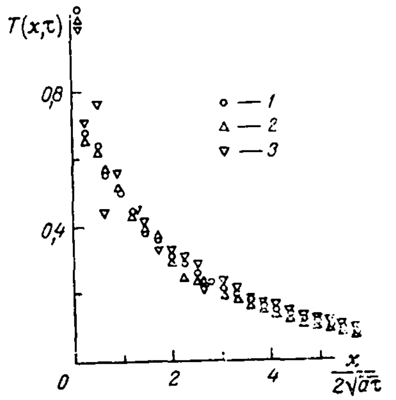

Fig. 1

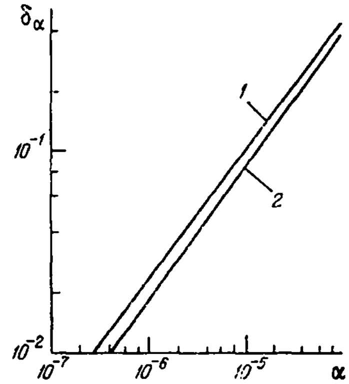

Fig. 2

Fig. 1. Family of recovered relationships for various noise levels. Data 1, 2, and 3 are for α values of −6.0, $-5\frac{2}{3}$, and $-5\frac{1}{3}$, respectively.

Fig. 2. Integral recovery error: 1) normal distribution; 2) uniform distribution.

1)). Using a program-injected coefficient (modulation index z_α), we regulated both distributions from 10^{-8} to 10^{-5}. A check carried out on vibratory functions showed that an increase in the index of modulation of the noise led to a one percent error in the result, starting with the value $z_\alpha \simeq 10^{-6}$.

Figure 1 displays a family of recovered relationships F(x) = T(x, τ) for the problem involving cooling of a semibounded body [2, 5]

$$\frac{\partial T}{\partial \tau} = a\frac{\partial^2 T}{\partial x^2}, \tag{2}$$
$$T(x,\ 0) = T_0,\ T(0,\ \tau) = T_c.$$

Solution of Eq. (2)

$$T(x,\ \tau) = (T_c - T_0)\,\mathrm{erfc}\left(\frac{x}{2\sqrt{a\tau}}\right) + T_0 \tag{3}$$

can be found on the basis of the image-function

$$\varphi(\lambda) = \frac{T_c - T_0}{\lambda}\exp\left(-x\sqrt{\frac{\lambda}{a}}\right) + \frac{T_0}{\lambda}. \tag{4}$$

Since the Laplace transform (4) is represented by means of an analytical expression, the function $\varphi(\lambda)$ was selected for numerical modeling, for the limits of which we used the function

$$f(\lambda) = \varphi(\lambda)(1 + z_\alpha),\ z_\alpha = z\cdot 10^{\alpha}, \tag{5}$$

where z is a random quantity produced by a standard generator. Figure 1 shows the high quality of recovery of T(x, τ) for the modulation index z_α with $\alpha \lesssim -7.0$. For further increase of the index ($\alpha \gtrsim -7.0$) the quality of the recovered function deteriorates.

As a rule, the stability of recovery of the function T(x, τ) falls with a decrease in argument x; this is apparently connected with the growth of the magnitude of the remainder term when the function (3) is expanded in basis functions of exponential type used in algorithm (1). The integral recovery error

$$\delta_\alpha = \left\{\int [T(x,\ \tau) - T_\alpha(x,\ \tau)]^2\,dx \Big/ \int T^2(x,\ \tau)\,dx\right\}^{1/2}, \tag{6}$$

describing the deviation of the desired function being recovered, for given z_α, from the unperturbed function (α = −8), is shown in Fig. 2. Parameters for the dependence shown in this figure were obtained by the method of least squares; approximately the integral error dependence in the case of function (3) can be represented, based on numerical modeling, in the form

$$\lg \delta_\alpha = 0{,}9\alpha + 4{,}2,\ \lg \delta_\alpha = 0{,}9\alpha + 4{,}0 \tag{7}$$

for uniform and normal distributions, respectively.

The expression for the recovery error

$$\delta_\alpha = 10^\alpha \left\langle \left\{ \frac{\int dx \left[\frac{\ln 2}{x} \sum_{n=1}^{N} z V_n f\left(\frac{n}{x} \ln 2\right)\right]^2}{\int dx \left[\frac{\ln 2}{x} \sum_{n=1}^{N} V_n f\left(\frac{n}{x} \ln 2\right)\right]^2} \right\}^{1/2} \right\rangle \tag{8}$$

is the mathematical expectation, i.e., the average over the bulk of realizations of the random quantity z. As the result of averaging, we find an approximate estimate from above for the desired mean

$$\delta_\alpha = \frac{1}{2} V_m \cdot 10^\alpha,\ V_m = \frac{1}{N} \sum_{n=1}^{N} |V_n| = 1{,}2 \cdot 10^5, \tag{9}$$

moreover, the dependences (7) and (9) do not contradict one another. We have thus established, in the case of the cooling problem (2), numerical characteristics relating the errors of the image function and the original function.

The dependence relations obtained point to the need for thorough preparation of the data for the Laplace transforms for algorithmic recovery of the original function. This can prove to be important in those cases in which the Laplace transform is, in turn, the result from the output of a correpsponding programmed algorithm, for example, in connection with the solution of a differential equation with an error stipulated by this algorithm.

NOTATION

x, coordinate, m; F(x), original function; n, number of basis functions; V_n, weighting coefficient; n, weighting coefficient index, $1 \le n \le N$; f(n/x ln 2), image function; z_α, stochastic modulation index; τ, current time, sec; T(x, τ), temperature field; $T_0 = T(x, 0)$; $T_c = T(0, \tau)$; a, thermal diffusivity, m²/sec; λ, Laplace transform variable; φ(λ), image function; f(λ), noise image function; z, random quantity with normalor rectangular distribution with mean zero and dispersion equal to 1; α, parameter characterizing the noise level; $T_\alpha(x, \tau)$, value of recovered temperature field in the presence of noise; δ_α, integral recovery error; V_m, mean value of weight coefficient moduli.

LITERATURE CITED

1. A. N. Tikhonov and V. Ya. Arsenin, Methods of Solution for Ill-Posed Problmems [in Russian], Moscow (1974).
2. N. M. Belyaev and A. A. Ryadno, Methods of the Theory of Thermal Conductivity [in Russian], Moscow (1982).
3. R. G. Jacquot, J. W. Steadman, and C. N. Rhodine, IEE Circuits Systems Mag., 5, No. 1, 4-8 (1983).
4. N. V. Kerov, Inzh.-Fiz. Zh., 49, No. 6, 966-970 (1985).
5. A. V. Lykov, Theory of Heat Conduction [in Russian], Moscow (1967).